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THE POTENTIAL FUNCTION.

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1. On p. 100, Vol. I. of THE ANALYST, Eq. (11), I have given the following value of the Potential Function:

$$V=\iiint\frac{\rho r'^2\sin\theta' dr' d\theta' d\omega'}{\{r^2+r'^2-2rr'[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\omega-\omega')]\}^{\frac{1}{2}}};\quad(1)$$

the integrations extending over the entire attracting mass. Now let

$$r'=cr,\text{ and } p=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\omega-\omega'),\ \ldots\ldots\quad(2)$$

and
$$R=(1+c^2-2cp)^{-\frac{1}{2}},\ \ldots\ldots\ldots\ldots\quad(3)$$

then
$$V=\iiint\frac{1}{r}\rho Rr'^2\sin\theta' dr' d\theta' d\omega'.\ \ldots\ldots\ldots\quad(4)$$

Let R be developed in to the following series:

$$R=1+cP_1+c^2P_2+c^3P_3+\ldots+c^iP_i+\ldots\quad(5)$$

P_i is a function of p independent of c. If we take the partial differential coefficients of (3) with respect to p we shall have

$$D_pR=c(1+c^2-2cp)^{-\frac{3}{2}}=cR^3,\ D_p^2R=3cR^2D_pR=3c^2R^5,\ D_p^3R$$
$$=3\,.\,5c^3R^7,\ D_p^4R=3\,.\,5\,.\,7c^4R^9,\ldots\ D_p^nR=3\,.\,5\,.\,7\ldots(2n-1)c^nR^{2n+1}.\quad(6)$$

Now let $R^{2n+1}=1+cP_1^{(n)}+c^2P_2^{(n)}+\ldots+c^iP_i^{(n)}+\ldots\quad(7)$

If we take the partial differential coefficients of (5) with respect to p, multiply (7) by $3\,.\,5\,.\,7\ldots(2n-1)c^n$, equate the coefficients of like powers of c, we shall find
$$D_p^nP_i=1\,.\,3\,.\,5\,.\,7\ldots(2n-1)P_{i-n}^{(n)}.\ \ldots\ldots\ldots\quad(8)$$

From this equation we can find $P_1^{(n)}$, $P_2^{(n)}$, &c. when P_1, P_2, &c. are known in terms of p.

2. From (3) we find, by taking partial differentials,

$$D_cR=-(c-p)R^3,\ D_c(c^2D_cR)=-D_c(c^3-c^2p)R^3=-(3c^2-2cp)R^3$$
$$-3(c^3-c^2p)R^2D_cR=-(3c^2-2cp)R^3+3c^2(c-p)^2R^5$$
$$D_pR=cR^3,\ D_p(p^2D_pR)=cD_p(p^2R^3)=2cpR^3+3c^2p^2R^5,\ D_p^2R=3c^2R^5.$$

From these equations we find, since $R^2(1+c^2-2cp)=1$,

$$D_c(c^2D_cR)+D_p^2R-D_p(p^2D_pR)=-c(3c-2p)(1+c^2-2cp)R^5$$
$$+3c^2(c-p)^2R^5+3c^2R^5-2cp(1+c^2-2cp)R^5-3c^2p^2R^5=0.$$

$$\therefore\quad D_c(c^2D_cR)+D_p[(1-p^2)D_pR]=0.\ \ldots\ldots\ldots\quad(9)$$

If we substitute the value of R given by (5) in this equation, and equate the coefficients of like powers of c, we shall find for the general value

$$i(i+1)P_i + D_p[(1-p^2)D_pP_i] = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

By means of this equation we are to calculate the value of P_i From (3) and (5) we have

$$D_cR = -\frac{c-p}{(1+c^2-2cp)^{\frac{3}{2}}} = P_1 + 2P_2c + \ldots + iP_ic^{i-1} + \ldots$$

$$(p-c)(1+P_1c+P_2c^2+\ldots+P_{i-1}c^{i-1}+\ldots) = (1+c^2-2cp)$$
$$\times(P_1+2P_2c+\ldots iP_ic^{i-1}+\ldots).$$

From this equation we have

$$iP_i = (2i-1)pP_{i-1} - (i-1)P_{i-2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If $i=2$, since $P_0=1$, and $P_1=p$, $2P_2=3p^2-1$; and for $i=3$, $3P_3 = 5pP_2 - 2P_1 = \frac{3}{2}(5p^3-3p)$.
From these equations we readily see that P_i will have this form

$$P_i = B_0p^i + B_1p^{i-2} + B_2p^{i-4} + \ldots + B_sp^{i-2s} + \ldots \qquad (12)$$

If we substitute the value of P_i given by (12) in (10) and equate the coefficients of like powers of p, we shall have

$$i(i+1)B_s + (i-2s+2)(i-2s+1)B_{s-1} - (i-2s)(i-2s+1)B_s = 0.$$

$$\therefore \qquad B_s = -\frac{(i-2s+2)(i-2s+1)}{2s(2i-2s+1)}B_{s-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Now make $s = 1, 2, 3,$ &c. in succession, and

$$B_1 = -\frac{i(i-1)}{2(2i-1)}B_0,\ B_2 = -\frac{(i-2)(i-3)}{4(2i-3)}B_1 = \frac{i(i-1)(i-2)(i-3)}{2.4(2i-1)(2i-3)}B_0$$
$$\text{\&c.} \quad . \quad . \quad . \quad . \quad (14)$$

We can in this way find all the coefficients in terms of B_0. We see that B_0 is the coefficient of c^ip^i in the development of R. We have

$$R = (1+c^2)^{-\frac{1}{2}}\left(1-\frac{2cp}{1+c^2}\right)^{-\frac{1}{2}} = (1+c^2)^{-\frac{1}{2}}\left[1 + \frac{2\lambda p}{2} + \frac{3.2^2\lambda^2p^2}{2.4}\ldots\right.$$
$$\left. + \frac{3.5\ldots(2i-1)2^i\lambda^ip^i}{2.4.6\ldots2i} + \ldots\right]. \quad \lambda = \frac{c}{1+c^2}$$

From this we see that the coefficient of c^ip^i is $\dfrac{1.3.5\ldots2i-1}{1.2.3\ldots i} = B_0 . . (15)$

$$\text{Hence } P_i = \frac{1.3.5\ldots2i-1}{1.2.3\ldots i}\left[p^i - \frac{i(i-1)}{2(2i-1)}p^{i-2} + \frac{i(i-1)(i-2)(i-3)}{2.4(2i-1)(2i-3)}p^{i-4}\right.$$
$$\left. - \ldots\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If we put aP_i for P_i in (10) the equation will still be satisfied; so that so long as a function of p differs from P only by having a constant multiplier, greater or less than unity, it will satisfy (10). The quantity P_i I shall call the p—coefficient of the ith order; and any other quantity as F_i, that will satisfy (10), I shall call the p—function of the ith order, i being any integer which denotes the highest power of p that enters in to the *coefficient* or the *function.*

If we should substitute for p its value given by (2), equation (10) would then be known as Laplace's Equation; and P_i would be called Laplace's Coefficient of the ith order; and F_i would be ealled Laplace's Function.

3. Let P_i and Q_i be any two p—functions. Equation (10) gives

$$i(i+1)\int_{-1}^{+1} P_i Q_n dp = -\int_{-1}^{+1} Q_n D_p[(1-p^2)D_p P_i]dp \quad \ldots . (17)$$

$$n(n+1)\int_{-1}^{+1} P_i Q_n dp = -\int_{-1}^{+1} P_i D_p[(1-p^2)D_p Q_n]dp \quad \ldots . (18)$$

$$i(i+1)\int_{-1}^{+1} P_i Q_n dp = -\Big[_{-1}^{+1} Q_n[(1-p^2)D_p P_i]\Big] + \int_{-1}^{+1} D_p P_i[(1-p^2)D_p Q_n]dp$$

$$= \Big[_{-1}^{+1} P_i[(1-p^2)D_p Q_n]\Big] - \int_{-1}^{+1} P_i D_p[(1-p^2)D_p Q_n]dp = n(n+1)\int_{-1}^{+1} P_i Q_n dp,$$

by (18). Therefore $[i(i+1) - n(n+1)]\int_{-1}^{+1} P_i Q_n dp = 0.$

So long as i differs from n, $i(i+1) - n(n+1)$ is not zero and therefore

$$\int_{-1}^{+1} P_i Q_n dp = 0. \quad \ldots \ldots \ldots \ldots \ldots (19)$$

This is a very important result. If $i = n$ it is indeterminate. For the case where $i = n$ we shall proceed as follows:

If we make $X = 1 - p^2$, then $D_p[(1-p^2)D_p P_i] = D_p(XD_p P_i)$. If this be differentiated m times we shall have

$$D_p^m(XD_p P_i) = D_p^m X D_p P_i + mD_p^{m-1} X D_p^2 P_i + \ldots + \frac{m(m-1)}{1\,.\,2} D_p^2 X D_p^{m-1} P_i$$

$$+ mD_p X D_p^m P_i + XD_p^{m+1} P_i = -m(m-1)D_p^{m-1} P_i - 2mpD_p^m P_i$$

$$+ (1-p^2)D_p^{m+1} P_i, \text{ since } X = 1 - p^2.$$

If we apply this to (10), we shall have, since $i(i+1) - m(m-1)$ $= (i-m+1)(i+m)$, by multiplying the resulting equation by $(1-p^2)^{m-1}$

$$D_p[(1-p^2)^m D_p^m P_i] + (i-m+1)(i+m)(1-p^2)^{m-1} D_p^{m-1} P_i = 0 \ldots (20)$$

Now multiply (20) by $D_p^{m-1}P_n$, and integrate the first term by parts, and

$$\int_{-1}^{+1} D_p^1[(1-p^2)^m D_p^m P_i] D_p^{m-1} P_n dp = \Big[(1-p^2)^m D_p^m P_i D_p^{m-1} P_n\Big]_{-1}^{+1}$$
$$- \int_{-1}^{+1} (1-p^2)^m D_p^m P_i D_p^m P_n \, dp$$

$$= -\int_{-1}^{+1} (1-p^2)^m D_p^m P_i D_p^m P_n dp$$
$$= -(i-m+1)(i+m)\int_{-1}^{+1}(1-p^2)^{m-1} D_p^{m-1} P_i D_p^{m+1} P_n dp\,;$$

or
$$\int_{-1}^{+1}(1-p^2)^m D_p^m P_i D_p^m P_n dp$$
$$= (i-m+1)(i+m)\int_{-1}^{+1}(1-p^2)^{m-1} D_p^{m\,1-} P_i D_p^{m-1} P_n dp. \;.\;.(21)$$

If we now make in succession $m = 1, 2, 3,$ &c., we shall have

$$\int_{-1}^{+1}(1-p^2) D_p P_i D_p P_n dp = i(i+1)\int_{-1}^{+1} P_i P_n dp, \qquad \int_{-1}^{+1}(1-p^2)^2 D_p^2 P_i D_p^2 P_n dp$$
$$= (i-1)(i+2)\int_{-1}^{+1}(1-p^2) D_p P_i D_p P_n dp = i(i-1)(i+1)(i+2)\int_{-1}^{+1} P_i P_n dp\,;$$

and finally $\int_{-1}^{+1} 1 - p^2)^m D_p^m P_i D_p^m P_n dp$

$$= (i-m+1)(i-m+2)\ldots i(i+1)\ldots(i+m)\int_{-1}^{+1} P_i P_n dp. \;.\;.(22)$$

Now make $n = i$ and $m = i$, and we have

$$\int_{-1}^{+1}(1-p^2)^i D_p^i P_i D_p^i P_i dp = 1\,.\,2\,.\,3\,.\,4\ldots 2i\int_{-1}^{+1} P_i^2 dp. \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;(23)$$

If we differentiate the value of P_i given by (16), i times, we shall have

$$D_p^i P_i = 1\,.\,3\,.\,5\ldots 2i-1. \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;(24)$$

This value in (23) gives

$$[1\,.\,3\,.\,5\ldots 2i-1]^2\int_{-1}^{+1}(1-p^2)^i dp = 1\,.\,2\,.\,3\,.\,4\ldots 2i\int_{-1}^{+1} P_i^2 dp,$$

or
$$\int_{-1}^{+1} P_i^2 dp = \frac{1\,.\,3\,.\,5\ldots 2i-1}{2\,.\,4\,.\,6\ldots 2i}\int_{-1}^{+1}(1-p^2)^i dp$$
$$= \frac{1\,.\,3\,.\,5\ldots 2i-1\,.\,2\,.\,4\,.\,6\ldots 2i}{2\,.\,4\,.\,6\ldots 2i\,.\,1\,.\,3\,.\,5\ldots 2i-1}\cdot\frac{2}{2i+1}.$$

$$\therefore \qquad \int_{-1}^{+1} P_i^2 dp = \frac{2}{2i+1}. \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;(25)$$

In this equation i is any integer. We can easily verify it for $i = 2$, since $P_2 = \frac{3}{2}(p^2 - \frac{1}{3})$, and also for $i = 3$, &c. The result expressed by (25) is also a very important one.

4. It is possible (granting that all differential equations of one variable are integrable) to arrange all algebraic functions of p, that do not become infinite between the limits of integration, into a series of p— functions, as will thus be seen. Let X be any algebraic function of p. In order that we may select from X what will make a p– function of the order i, say F_i, it is only necessary to find the coefficient which corresponds to what we have represented by B_0, (15), in the coefficients; for the *law* of the terms of the function is fixed, being the same as in (16). Let us call the required coefficient $A_0^{(2)}$, then it will be evident by (25) that

$$B_0 \int_{-1}^{+1} P_i F_i dp = \frac{2A_0^{(i)}}{2i+1}; \quad \ldots\ldots\ldots\ldots \quad (26)$$

or,

$$B_0 \int_{-1}^{+1} P_i X dp = \frac{2A_0^{(i)}}{2i+1}; \quad \ldots\ldots\ldots\ldots \quad (27)$$

since by (19) all the terms of X not required to form F_i, will disappear. If X be developed according to the positive powers of p, then we may make

$$X = A_0^{(i)}\left[p^i - \frac{i(i+1)}{2(2i-1)}p^{i-2} + \ldots\right] + A_0^{(i-1)}\left[p^{i-1} - \frac{(i-1)(i-2)}{2(2i-3)}p^{i-3} + \ldots\right] + \ldots$$

and compare the coefficients of like powers of p. Let

$X = p^3 + p^2 + p + 1 = A_0^{(0)} + A_0^{(1)}p + A_0^{(2)}(p^2 - \frac{1}{3}) + A_0^{(3)}(p^3 - \frac{3}{5}p)$.

Then $\quad A_0^{(3)} = 1, \quad A_0^{(2)} = 1, \quad A_0^{(1)} = \frac{8}{5}, \quad A_0^{(0)} = \frac{4}{3}$.

By (27) $\int_{-1}^{+1}(p^3 + p^2 + p + 1)dp = 2A_0^{(0)} = 2(\frac{1}{3} + 1),\ A_0^{(0)} = \frac{4}{3}$,

$\int_{-1}^{+1}(p^4 + p^3 + p^2 + p)dp = \frac{2}{3}A_0^{(1)} = 2(\frac{1}{5} + \frac{1}{3}),\ A_0^{(1)} = \frac{8}{5}$,

$\frac{9}{4}\int_{-1}^{+1}(p^2 - \frac{1}{3})(p^3+p^2+p+1)dp = \frac{2}{5}A_0^{(2)} = \frac{9}{2}(\frac{1}{5}+\frac{1}{3}-\frac{1}{9}-\frac{1}{3}),\ A_0^{(2)} = 1$; &c.;

since $P_0 = 1$, $P_1 = p$, $P_2 = \frac{3}{2}(p^2 - \frac{1}{3})$, $P_3 = \frac{5}{2}(p^3 - \frac{3}{5}p)$, &c.

Since the quantity within the brackets in (16) must be the same for all these functions, it is evident that if X is a surd, as $\sqrt{(1 + p^2)}$, the number of functions is infinite. Now let $X = A_0 + A_1p^2$, and we shall find two functions of the order 0 and 2, as follows;

$$A_0 + A_1p^2 = (A_0 + \tfrac{1}{3}A_1) + A_1(p^2 - \tfrac{1}{3}). \quad \ldots\ldots \quad (28)$$

5. Let us now make an application of the principles which we have demonstrated, to find the potential of an oblate spheroid for an external point situated in the prolongation of the axis of revolution, the density being homogeneous.

From (2), (4), and (5) we have

$$V=\int_{-1}^{+1}\int_{0}^{r'}\int_{0}^{2\pi}\rho r'^2dr'dpd\omega'\left[\frac{1}{r}+P_1\frac{r'}{r^2}+P_2\frac{r'^2}{r^3}+\ldots+P_i\frac{r'^i}{r^{i+1}}+\right]$$

by making $\sin\theta'd\theta' = -dp$ (as it evidently is), and changing the sign of V. Since ω' is independent of r' and p, and the first term evidently the mass of the spheroid divided by r (equal to $m\div r$,)

$$V=\frac{m}{r}+2\pi\rho\int_{-1}^{+1}dp\left[\tfrac{1}{4}P_1\frac{r'^4}{r^2}+\tfrac{1}{5}P_2\frac{r'^5}{r^3}+\ldots+\frac{1}{i+3}P_i\frac{r'^{i+3}}{r^{i+1}}+\ldots\right]\ldots(29)$$

Let $r'=\dfrac{a\sqrt{(1-e^2)}}{\sqrt{[1-e^2(1-p^2)]}}$, and let $\dfrac{r'^{i+3}}{a^{i+3}}$ be developed in to a series of p–functions, so that

$$r'^{i+3}=a^{i+3}[F_0+F_1+F_2+\ldots+F_i+\ldots]\ldots(30)$$

If this be substituted in (29) we see by (19) that every term, when integrated, except the one containing P_i will disappear. If we expand the value of r'^{i+3}, retaining only e^2, we shall find

$$r'^{i+3}=a^{i+3}\left[1-\frac{i+3}{2}e^2p^2\right]=a^{i+3}\left[\left(1-\frac{i+3}{6}e^2\right)-\frac{i+3}{2}e^3\left(p^2-\tfrac{1}{3}\right)\right]$$
$$=a^{i+3}[F_0+F_2].$$

From this we see that $i=0$ and $i=2$ are the only values to be used; and since there is no P_0 in (29), we have

$$V=\frac{m}{r}-\pi\rho\frac{a^5}{r^3}\int_{-1}^{+1}e^2P_2(p^2-\tfrac{1}{3})dp=\frac{m}{r}-\pi e^2\rho\frac{a^5}{r^3}\int_{-1}^{+1}\tfrac{3}{2}(p^2-\tfrac{1}{3})^2dp.$$

$$\therefore\qquad V=\frac{m}{r}-\frac{4}{15}\pi e^2\rho\frac{a^5}{r^3}=\frac{m}{r}-\frac{ma^2e^2}{5r^3}.\ \ldots\ldots\ (31)$$

The preceding discussion will help the student to understand the nature and uses of Laplace's *Coefficients* and *Functions* in their more general form as given in works on the figure of the earth and elsewhere. Some mathematical expressions contain curious properties.

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Gauss, C. F. Werke. Band VI. Herausgegeben von der koniglichen Gesellschaft der Wissenschaften zu Gottingen. Gottingen. 1874. 4to. 664 pp. 25 M.

Reuschle, C. G. Tafeln complexer Primzahlen, welche aus Wurzeln der Einheit gebildet sind. Auf dem Grunde der Kummerschen Theorie der complexen Zahlen berechnet. Berlin. 1875. 4to. VII. 671 pp. 24M.